JOURNAL OF COMBINATORIAL THEORY, Series B 47, 120–123 (1989)

Note

On Local Structure of a Distance-Regular Graph of Hamming Type

KAZUMASA NOMURA

Tokyo Ikashika University, Kounodai, Ichikawa 272, Japan

Communicated by U. S. R. Murty

Received August 11, 1987

We shall prove a result about local structures of distance-regular graphs with $c_2=2$, $c_3=3$. This gives a generalization and simple proof of the essential part of Egawa's characterization of the Hamming scheme [*J. Combin. Theory Ser. A* **31** (1981), 108–125]. © 1989 Academic Press, Inc.

Let G be a distance-regular graph with the vertex set V and let ∂ denote the metric on V. For $u, v \in V$ and non-negative integers r, s, we define

$$\Gamma_r(u) = \{x \in V \mid \partial(x, u) = r\}$$

$$D_s^r(u, v) = \Gamma_r(u) \cap \Gamma_s(v).$$

Let a_i, b_i, c_i (($0 \leqq i \leqq d$, d is the diameter of G) be the usual intersection numbers of G. Definitions and precise descriptions about distance-regular graphs will be found in [1]. For subsets X, Y of V, the number of edges between X and Y will be denoted by $e(X, Y)$, or $e(x, Y)$ if $X = \{x\}$ is a singleton.

In this paper we consider distance-regular graphs with $c_2=2$ and $c_3=3$. The Hamming scheme $H(n, q)$ $(n>2)$ gives an example of this type. Egawa proved a characterization theorem of the Hamming scheme in [3]. The following results give a simple proof of Egawa's theorem $(q \neq 4)$ in a more general situation. Theorem 1 will be useful in the classification of distance-regular graphs with $c_2=2$, $c_3=3$.

In the following we assume G is a distance-regular graph with $c_2=2$, $c_3=3$.

THEOREM 1. *Let u, v be vertices in G with $\partial(u, v)=3$. Then the number of edges in $D_2^1(u, v)$ is at most one.*

0095-8956/89 $3.00

THEOREM 2. *Let uv be an edge in G and let $P_1, ..., P_m$ be the connected components in $D_1^1(u, v)$, $p_i = |P_i|$. Then*

$$a_2 \geqq a_1 - 1 + \sum_{i=1}^{m} p_i(a_1 - p_i).$$

THEOREM 3. *If $a_1 > 2$* and *$a_2 = 2a_1$, $D_1^1(u, v)$ is a clique for each edge uv in G.*

Remark 1. Clearly $H(n, q)$ satisfies the assumption of the above theorem. One of the essential parts of [3] is to show that $D_1^1(u, v)$ is a clique (See [3, Proposition 4.1]).

Remark 2. The assumption $a_2 = 2a_1$ can be replaced by $a_2 \leqq 2a_1$. But there is not much point in doing that since $a_2 \geqq 2a_1$ holds if $c_2 = 2$, $c_3 = 3$.

In the following proof we shall use intersection diagrams. Definitions and elementary properties of intersection diagrams are described in [2].

Proof of Theorem 1. We have $|D_2^1(u, v)| = |D_1^2(u, v)| = c_3 = 3$. Let $D_2^1(u, v) = \{x_1, x_2, x_3\}$, $D_1^2(u, v) = \{y_1, y_2, y_3\}$. From $c_2 = 2$, we have $e(x_i, D_1^2(u, v)) = e(y_i, D_2^1(u, v)) = 2$ $(i = 1, 2, 3)$. We may assume $x_i \operatorname{adj} y_i$, $x_2 \operatorname{adj} y_i$, $y_2 \operatorname{adj} x_i$ for $i = 1, 3$.

First assume there are just two edges in $D_2^1(u, v)$, say x_1x_2 and x_2x_3. Then $\partial(x_1, x_3) = 2$. But then $\{u, x_2, y_2\} \subset D_1^1(x_1, x_3)$. This is impossible since $c_2 = 2$.

In the following we assume $D_2^1(u, v)$ is a clique. If $D_1^2(u, v)$ contains an edge, say y_1y_2, then we get $\partial(y_1, x_3) = 2$ and $\{y_2, x_1, x_2\} \subset D_1^1(y_1, x_3)$, a contradiction. So there is no edge in $D_1^2(u, v)$.

Now consider the intersection diagram with respect to the edge vy_1. Put $D_s^r = D_s^r(v, y_1)$. Then $y_2, y_3 \in D_2^1$, $x_1, x_2 \in D_1^2$, $x_3 \in D_2^2$, $u \in D_2^3$. By [2, Lemma 4], there is an edge x_3w with $w \in D_3^2$. Then $D_2^1(y_1, w) = \{v, x_1, x_2\}$ and $x_3 \in D_1^2(y_1, w)$. Let z_1, z_2 be the vertices of $\Gamma_1(w) \cap D_2^1$. Then $D_1^2(y_1, w) = \{x_3, z_1, z_2\}$ and we may assume $z_1 \operatorname{adj} x_1$, $z_2 \operatorname{adj} x_2$ as before. Since $e(x_1, D_2^1) = 1$, we get $z_1 = y_2$. Similarly $z_2 = y_3$. Then y_2x_3 and y_3x_3 are edges in $D_1^2(y_1, w)$. This implies $y_2 \operatorname{adj} y_3$ since the number of edges in $D_1^2(y_1, w)$ is not 2. This contradicts the fact that $D_1^2(u, v)$ has no edge. ∎

Proof of Theorem 2. If $m \leqq 1$ the conclusion holds since $a_2 \geqq a_1 - c_2 + 1$ holds generally by $a_1 + b_1 + c_1 = a_2 + b_2 + c_2$ and $b_1 \geqq b_2$. So we assume $m \geqq 2$. We consider the intersection diagram with respect to the edge uv and put $D_s^r = D_s^r(u, v)$.

We remark that each P_i is a clique. For if $x_1x_2x_3$ is a 2-path in D_1^1, we have $\{u, v, x_2\} \subset D_1^1(x_1, x_3)$. This implies $\partial(x_1, x_3) \neq 2$. Hence P_i is a clique. Remark also that $e(x, D_1^2) = a_1 - p_i > 0$ holds for $x \in P_i$ since $e(x, \Gamma_1(v)) = a_1$ and $\Gamma_1(v) = \{u\} \cup D_1^1 \cup D_1^2$.

Now we divide D_1^2 into two subsets,

$$A=\{x\in D_1^2 \mid e(x, D_1^1)=1\}, \qquad B=\{x\in D_1^2 \mid e(x, D_2^1)=1\}.$$

We have $D_1^2=A\cup B$, $A\cap B=\varnothing$ since $c_2=2$. We have also $A\neq\varnothing$ by the above remark. Take $y\in A$ and let x be the vertex in $\Gamma_1(y)\cap D_1^1$. Put $F=\Gamma_1(y)\cap D_2^3$.

For $z\in F$ we have $|\Gamma_1(z)\cap D_1^2|=2$, and let w be the vertex in $\Gamma_1(z)\cap D_1^2$ with $w\neq y$. We claim that $w\in B$. If $w\in A$ then there is an edge wx' with $x'\in D_1^1$. But then $D_2^1(u, z)=\{v, x, x'\}$ with $\partial(v, x)=\partial(v, x')=1$, contradicting Theorem 1. So we have $w\in B$. Thus we get a mapping $f\colon F\to B$ with $f(z)\in\Gamma_1(z)$.

We claim f is injective. Take $z_1, z_2\in F$ with $z_1\neq z_2$ and assume $f(z_1)=f(z_2)=w$. Let wx' be the edge with $x'\in D_2^1$. Then $D_2^1(u, z_1)=D_2^1(u, z_2)=\{v, x, x'\}$. Take $w_i\in D_1^2(u, z_i)$ with $w_i\neq y$, $w_i\neq w$ $(i=1, 2)$. Then we have $\{u, w_1, w_2\}\subset D_1^1(x, x')$ by considering the subgraph $\bigcup_{j=0}^{3} D_{3-j}^{j}(u, z_i)$. This implies $w_1=w_2$ since $\partial(x, x')=2$ and $c_2=2$. But then $|D_1^1(y, w)|$, $|D_1^1(w, w_1)|\geqq 3$ and hence $\partial(y, w)=\partial(w, w_1)=1$, contradicting Theorem 1. Thus f is injective. In particular we get $|F|\leqq|B|$.

Next we count $|A|$. Since $e(y, D_1^1)=1$ for $y\in A$, we have $|A|=e(A, D_1^1)=e(D_1^2, D_1^1)$. On the other fhand we have $e(x, D_1^2)=a_1-p_i$ for $x\in P_i$. Thus

$$|A|=\sum_{x\in D_1^1} e(x, D_1^2)=\sum_{i=1}^{m}\sum_{x\in P_i}(a_1-p_1)=\sum_{i=1}^{m}p_i(a_1-p_i).$$

Then Theorem 2 follows from $b_2=|F|\leqq|B|$, $|D_1^2|=b_1$, and $a_1+b_1+c_1=a_2+b_2+c_2$. ∎

Remark 3. We claim that there is no edge between A' and B' if the mapping f is surjective in the above proof where

$$A'=\{x\in D_2^1 \mid e(x, D_1^1)=1\}, \qquad B'=\{x\in D_2^1 \mid e(x, D_1^2)=1\}.$$

To show this fact, we assume there is an edge $a'b'$ with $a'\in A'$, $b'\in B$. Take edges $a'x$ and xy with $x\in D_1^1$, $y\in D_1^2$. Let $b'b$ be the edge with $b\in B$. Since f is surjective, there is a vertex $z\in F$ with $y \operatorname{adj} z$, $z \operatorname{adj} b$. Then $D_2^1(u, z)=\{v, x, b'\}$, $\{y, b\}\subset D_1^2(u, z)$. So there is one more vertex w in $D_1^2(u, z)$. Then w is adjacent to x and b'. But then there are three vertices u, a', w which are adjacent to both x and b'. This is impossible since $\partial(x, b')=2$.

We remark also that the surjectivity of f does not depend on the choice of y.

Proof of Theorem 3. We shall use the notation of Theorem 2. We need to show $m=1$. Since $a_2=2a_1$, we get

$$a_1+1 \geqq \sum_{i=1}^{m} p_i(a_1-p_i).$$

Since $\sum_{i=1}^{m} p_i=a_1$, we get a contradiction when $m \geqq 3$. So we assume $m=2$. Then the above inequality becomes

$$p_1+p_2+1 \geqq 2p_1p_2.$$

This implies $p_1+p_2 \leqq 3$. Hence we get $a_1=3$ by the asumption $a_1>2$. Note that equality holds in the above inequality in this case. Therefore the mapping f in the proof of Theorem 2 should be surjective. So there is no edge between A' and B' in the above remark. Note that $e(a', D_2^1)=a_1-1=2$ holds for every vertex a' in A', and hence A' is 2-regular of size 4, that is a 4-cycle x_1, x_2, x_3, x_4. But then $\partial(x_1, x_3)=2$ and $\{x_2, x_4, u\} \subset D_1^1(x_1, x_3)$, contradicting $c_2=2$. ∎

References

1. E. Bannai and T. Ito, "Algebraic Combinatorics I," Benjamin, New York, 1984.
2. A. Boshier and K. Nomura, A remark on the intersection array of a distance-regular graph, *J. Combin. Theory Ser. B* **44** (1988), 147–153.
3. Y. Egawa, Characterization of $H(n, q)$ by the parameters, *J. Combin. Theory Ser. A* **31** (1981), 108–125.